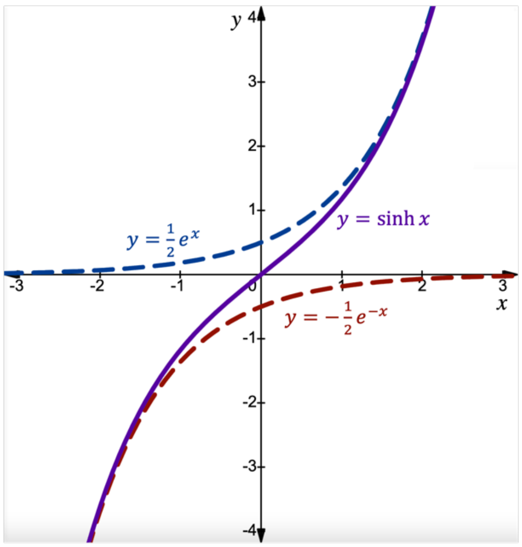
Hyperbolic Function - Definitions, Identities, & More

Definitions, Pronunciations, Graphs, Domains, & Ranges



Hyperbolic Sine

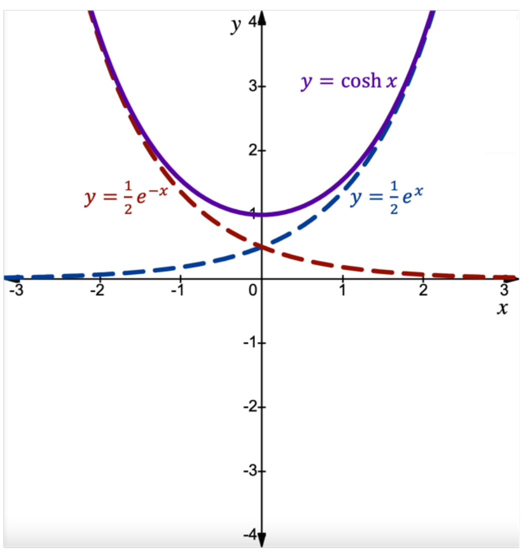
$y = \sinh x$

Definition: $\sinh x = \frac{e^x - e^{-x}}{2}$

sinh is pronounced *sinch*

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$



Hyperbolic Cosine

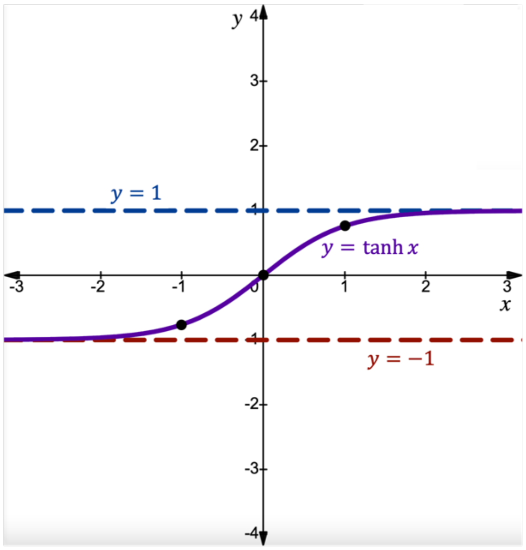
$y = \cosh x$

Definition: $\cosh x = \frac{e^x + e^{-x}}{2}$

cosh is pronounced *cosh*

Domain: $(-\infty, \infty)$

Range: $[1, \infty)$



Hyperbolic Tangent

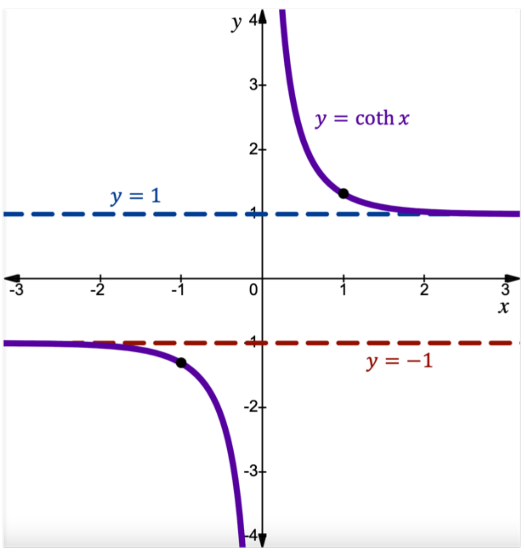
$y = \tanh x$

Definition: $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

tanh is pronounced *tanch*

Domain: $(-\infty, \infty)$

Range: $(-1,1)$



Hyperbolic Cotangent

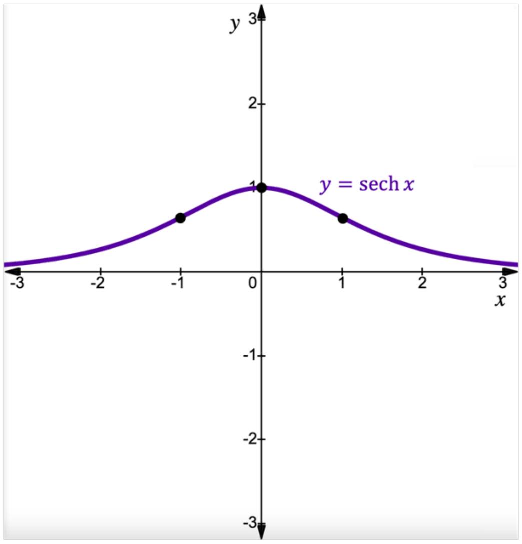
$y = \coth x$

Definition: $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

cotanh is pronounced *cotanch*

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, -1) \cup (1, \infty)$



Hyperbolic Secant

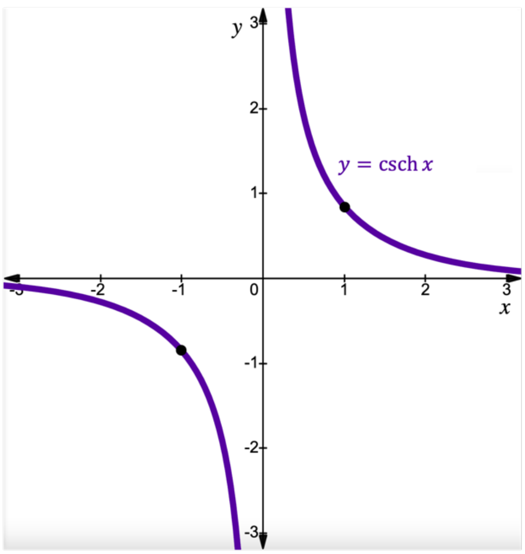
$y = \operatorname{sech} x$

Definition: $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

csch is pronounced *seech* *

Domain: $(-\infty, \infty)$

Range: $(0, 1]$



Hyperbolic Cosecant

$y = \operatorname{csch} x$

Definition: $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

csch is pronounced *coseech* *

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Identities Involving Hyperbolic Functions

- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \operatorname{sech}^2 x$
- $\coth^2 x - 1 = \operatorname{csch}^2 x$
- $\cosh(-x) = \cosh x$
- $\sinh(-x) = -\sinh x$
- $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
- $\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$
- $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
- $\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$

Many other identities can be derived from these identities.

Derivatives of the Hyperbolic Functions

- $\frac{d}{dx}\sinh x = \cosh x$
- $\frac{d}{dx}\cosh x = \sinh x$
- $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
- $\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$
- $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech}(x)\tanh(x)$
- $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch}(x)\coth(x)$

Analogous to Derivatives of the Trig Functions

Did you notice that the derivatives of the hyperbolic functions are analogous to the derivatives of the trigonometric functions, except for some differences in sign? Once again the derivative of the cofunction is the cofunction of the derivative (except possibly for the sign). How are the signs different? When we list the hyperbolic in this order, the first three derivatives are positive and the last three are negative.

Formulas for the Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the inverse hyperbolic functions.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right),\ x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right),\ x \in (1, \infty)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ x \in (-1,1)$$

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ x \in (-\infty, -1) \cup (1, \infty)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{x+\sqrt{1-x^2}}{x}\right),\ x \in (0, 1]$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right),\ x \in (-\infty, 0) \cup (0, \infty)$$

See the next page for the derivative formulas for the inverse hyperbolic functions.

Derivatives of the Inverse Hyperbolic Functions

Looking at the derivative formulas below, we could mistakenly conclude that $\frac{d}{dx}\tanh^{-1}x$ and $\frac{d}{dx}\coth^{-1}x$ are equivalent. But they are not! The domain of $\tanh^{-1}x$ $x \in (-1,1)$. And the domain of $\coth^{-1}x$ is $x \in (-\infty, 0) \cup (0, \infty)$. So, their respective derivative functions have no x values in common.

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}},\ x \in \mathbb{R}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}},\ x \in (1, \infty)$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2},\ x \in (-1,1)$$

$$\frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2},\ x \in (-\infty, -1) \cup (1, \infty)$$

$$\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{-1}{x\sqrt{1-x^2}},\ x \in (0,1)$$

$$\frac{d}{dx}\operatorname{csch}^{-1}x = \frac{-1}{|x|\sqrt{x^2+1}},\ x \in (-\infty, 0) \cup (0, \infty)$$